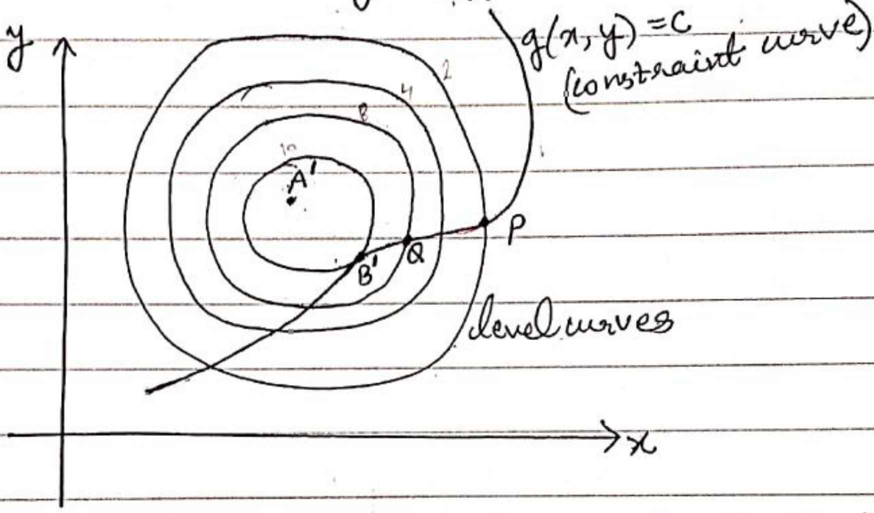
Ch-18 Constrained Optimization

(Leave out sections 18.8, 18.9, 18.10)

Two Variables, One Equality Constraint

Suppose the problem is max. $f(x, y)$

s.t $g(x, y) = c$



$A' \rightarrow$ unconstrained max^m pt. of f

The closer a level curve of f is to point A', the higher is the value of f along that level curve.

$B' \rightarrow$ constrained max^m pt. of f

At B', constraint curve touches (without intersecting) a level curve for f.

$\Rightarrow$ Slope of tangent to the curve $g(x, y) = c$ = Slope of tangent to the level curve of f.

$$-\frac{g_1'(x,y)}{g_2'(x,y)} = -\frac{f_1'(x,y)}{f_2'(x,y)}$$

or $$\boxed{\frac{f_1'(x,y)}{f_2'(x,y)} = \frac{g_1'(x,y)}{g_2'(x,y)}}$$

Minimizing $f(x,y)$ s.t $g(x,y) = c$ gives the same condition.

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The Lagrangean Multiplier Method

Problem : max (min) $f(x, y)$ s.t $g(x,y) = c$

$\boxed{\text{Steps}}$

① Write down the Lagrangean function

$$\mathcal{L}(x,y) = f(x,y) - \lambda\left(g(x,y) - c\right)$$

or $$\mathcal{L}(x,y) = f(x,y) + \lambda\left(c - g(x,y)\right)$$

where λ is a constant called **Lagrangean Multiplier**

② $$\mathcal{L}_1'(x,y) = f_1'(x,y) - \lambda g_1'(x,y) = 0$$

$$\mathcal{L}_2'(x,y) = f_2'(x,y) - \lambda g_2'(x,y) = 0$$

③ $$\left.\begin{aligned} f_1'(x,y) &= \lambda g_1'(x,y) \\ f_2'(x,y) &= \lambda g_2'(x,y) \\ g(x,y) &= c \end{aligned}\right]$$ Necessary conditions for (x,y) to solve the given problem λ

④ Solve these 3 eqns for the 3 unknowns x, y & λ.

Economic Interpretation of λ

max. $f(x,y)$ s.t $g(x,y) = c$

Suppose x^* & y^* are the values of x & y that solve this problem.
In general, x^* & y^* depend on c.
We assume that $x^* = x^*(c)$ & $y^*(c)$ & $\lambda = \lambda(c)$.

Thus, $\underbrace{f^*(c)}_{\text{Optimal value function}} = f(x^*(c), y^*(c))$

① Results →

$$\boxed{\frac{d f^*(c)}{dc} = \lambda(c)}$$

Thus $\lambda = \lambda(c)$ is the rate at which the optimal value of the objective function changes w.r.t ~~of~~ changes in c.

In economic applications, 'c' often denotes the available stock of some resource & λ ~~denotes~~ is called shadow price of the resource. $f(x, y)$ denotes utility or profit.

② If dc is small, $\Delta f^*(c) \approx d f^*(c)$

$$\boxed{f^*(c+dc) - f^*(c) \approx \lambda(c)\, dc}$$

Q- Consider the problem

$$\min f(x, y) = x^2 + y^2 \text{ s.t } x + 2y = a \text{ (a is a const)}$$

(a) Solve the constrained optimization problem by reducing it to an unconstrained one.

$$x = a - 2y$$

$$g(y) = (a-2y)^2 + y^2 = a^2 + 5y^2 - 4ay$$

$$\frac{dg}{dy} = 10y - 4a = 0 \Rightarrow y = \frac{2a}{5}$$

$$\frac{d^2 g}{dy^2} = 10 > 0 \Rightarrow y = \frac{2a}{5} \text{ is a min}^m \text{ pt. of } g.$$

$$x = a - \frac{4a}{5} = \frac{a}{5}$$

Min^m pt. of f $\left(\frac{a}{5}, \frac{2a}{5}\right)$

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(b) Solve the given problem by the Lagrangean method.

$$\mathcal{L}(x,y) = x^2+y^2-\lambda(x+2y-a)$$

$$\mathcal{L}_1'(x,y) = 2x-\lambda = 0 \Rightarrow 2x=\lambda$$

$$\mathcal{L}_2'(x,y) = 2y-2\lambda = 0 \Rightarrow y=\lambda$$

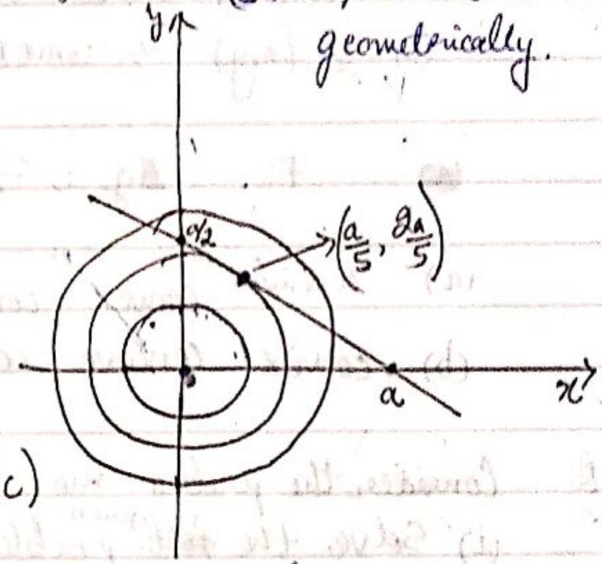
$$x+2y=a$$

$$\frac{\lambda}{2}+2\lambda = a$$

$$5\lambda = 2a \Rightarrow \boxed{\lambda = 2a/5}$$

$$\boxed{x = \frac{a}{5}},\ \boxed{y = 2a/5}$$

(c) Explain the solution geometrically.



(c) Verify the eqn $\frac{df^*(c)}{dc} = \lambda(c)$

$$f^*(a) = (x^*)^2+(y^*)^2 = \frac{a^2}{25}+\frac{4a^2}{25} = \frac{5a^2}{25} = \frac{a^2}{5}$$

$$\frac{df^*(a)}{da} = \frac{2a}{5} = \lambda(a) \quad \text{(Verified)}$$

18.3 (Only the statement of Thm 18.1 is to be done)

Thm 18.1 Lagrange's Theorem

Suppose that $f(x,y)$ & $g(x,y)$ have continuous partial derivatives in a domain A of the xy-plane, & that (x_0,y_0) is both an interior pt. of A & a global local extreme pt. for $f(x,y)$ s.t to the constraint $g(x,y)=c$. Suppose further that $g_1'(x_0,y_0)$ & $g_2'(x_0,y_0)$ are not both 0. Then there exists a unique no. λ s.t the Lagrangean function $\mathcal{L}(x,y) = f(x,y)-\lambda(g(x,y)-c)$ has a stationary point at (x_0,y_0) i.e. $\mathcal{L}_1'(x_0,y_0)=0$ & $\mathcal{L}_2'(x_0,y_0)=0$.

$$\max(\min)\ f(x,y) \text{ s.t } g(x,y)=c \quad —①$$

Th^{m} 18.2 Global Sufficiency – Suppose that $f(x,y)$ & $g(x,y)$ in problem ① have continuous partial derivatives of the first order / are continuously differentiable functions on an open convex set A in R^2, & let $(x_0,y_0)\in A$ be an interior ~~pt.~~ stationary pt. for the Lagrangean function $\mathcal{L}(x,y)= f(x,y)-\lambda\,(g(x,y)-c)$
Suppose that $g(x_0,y_0)=c$. Then

(a) $\mathcal{L}(x,y)$ is concave $\Rightarrow (x_0,y_0)$ solves the maximization problem in ①

(b) $\mathcal{L}(x,y)$ is convex $\Rightarrow (x_0,y_0)$ solves the minimization problem in ①.

$$f \qquad g \qquad -g \qquad \mathcal{L} = f(x,y)+[-\lambda g(x,y)]+\lambda c$$

(a) concave convex concave $\Rightarrow$ concave

(b) convex concave convex $\Rightarrow$ convex

Q– Consider the problem $\max.\ 10x^{1/2}y^{1/3}$ s.t $2x+4y=m$ $(x>0, y>0)$

(a) Solve the ~~foll.~~ given problem by the Lagrangean method.

(b) Prove that you have found the optimal solution.

(a) $$\mathcal{L}(x,y) = 10x^{1/2}y^{1/3}-\lambda\,(2x+4y-m)$$

$$\mathcal{L}_1'(x,y) = 10\,\frac{1}{2}\cdot\frac{y^{1/3}}{\sqrt{x}}-2\lambda = 0 \Rightarrow 5\,\frac{y^{1/3}}{\sqrt{x}}-2\lambda=0 \quad —①$$

$$\mathcal{L}_2'(x,y) = 10\sqrt{x}\,\frac{1}{3\cdot y^{2/3}}-4\lambda=0 \Rightarrow \frac{5}{3}\,\frac{\sqrt{x}}{y^{2/3}}-2\lambda=0 \quad —②$$

$$2x+4y=m \quad —③$$

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$$\frac{\cancel{5}\, y^{1/3}}{\sqrt{x}} = \frac{5}{3}\frac{\sqrt{x}}{y^{2/3}}$$

$$\boxed{x = 3y}$$

Put in ③, $10y = m \Rightarrow \boxed{y = m/10}$, $\boxed{x = 3m/10}$

$$① \Rightarrow \lambda = \frac{5}{2}\frac{y^{1/3}}{\sqrt{x}} = \frac{5}{2}\frac{\left(\frac{m}{10}\right)^{1/3}}{\left(\frac{3m}{10}\right)^{1/2}} = \cancel{2.5\left(\frac{m}{10}\right)^{1/3}\left(\frac{10}{3m}\right)^{1/2}}$$

$$= 2.5\left(\frac{10}{27m}\right)^{1/6}$$

$$= \frac{2.5}{\sqrt{3}} m^{-1/6} 10^{1/6}$$

(b) $(x_0^*, y_0^*) = \left(\frac{3m}{10}, \frac{m}{10}\right)$

$$f(x,y) = 10x^{1/2}y^{1/3}$$

Second derivative Test for concavity (Th^m 17.9)

$$F_{11}'' \le 0,\ F_{22}'' \le 0,\ \begin{vmatrix} F_{11}'' & F_{12}'' \\ F_{21}'' & F_{22}'' \end{vmatrix} \ge 0 \Rightarrow f \text{ is } \underline{\text{concave}}.$$

$$F_1' = 10\left(\frac{1}{2}\right)x^{-1/2}y^{1/3} = 5x^{-1/2}y^{1/3}$$

$$F_{11}'' = 5\left(\frac{-1}{2}\right)x^{-3/2}y^{1/3} = \frac{-5}{2}x^{-3/2}y^{1/3} < 0$$

$$F_{12}'' = 5x^{-1/2}\left(\frac{1}{3}\right)y^{-2/3} = F_{21}''$$

$$F_2' = 10x^{1/2}\frac{1}{3}y^{-2/3}$$

$$F_{22}'' = 10x^{1/2}\frac{1}{3}\left(\frac{-2}{3}\right)y^{-5/3} = \frac{-20}{9}x^{1/2}y^{-5/3} < 0$$

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$$\begin{vmatrix} -\frac{5}{2} x^{-3/2} y^{1/3} & \frac{5}{3} x^{-1/2} y^{-2/3} \\ \frac{5}{3} x^{-1/2} y^{-2/3} & -\frac{20}{9} x^{1/2} y^{-5/3} \end{vmatrix}$$

$$= \frac{50}{9} x^{-1} y^{-4/3} - \frac{25}{9} x^{-1} y^{-4/3}$$

$$= \frac{25}{9xy^{4/3}} > 0$$

$\Rightarrow F(x,y)$ is concave

$g(x,y) = 2x + 4y$ (linear function is convex as well as concave)

$\Rightarrow \mathcal{L}(x,y)$ is concave $\Rightarrow (x_0, y_0)$ solves the maximization problem.

Sufficient Conditions for Local Extreme Pts.

local max(min) $f(x,y)$ s.t $g(x,y) = c$ — ①

Suppose (x_0, y_0) is a stationary pt. of $\mathcal{L}(x,y)$

(a) $D(x_0, y_0) > 0 \Rightarrow (x_0, y_0)$ solves the local maximization problem ①

(b) $D(x_0; y_0) < 0 \Rightarrow (x_0, y_0)$ solves the local minimization problem ①

$$D(x,y) = \begin{vmatrix} 0 & g_1'(x,y) & g_2'(x,y) \\ g_1'(x,y) & \mathcal{L}_{11}''(x,y) & \mathcal{L}_{12}''(x,y) \\ g_2'(x,y) & \mathcal{L}_{21}''(x,y) & \mathcal{L}_{22}''(x,y) \end{vmatrix}$$

Bordered Hessian Determinant

$$= \begin{vmatrix} 0 & g_1' & g_2' \\ g_1' & f_{11}'' - \lambda g_{11}'' & f_{12}'' - \lambda g_{12}'' \\ g_2' & f_{21}'' - \lambda g_{21}'' & f_{22}'' - \lambda g_{22}'' \end{vmatrix}$$

2016 Q5(c)
2015 Q5(a).

Q- ~~local max (min) $f(x,y) = x^2+y^2$~~

Show that $\left(\frac{a}{5}, \frac{2a}{5}\right)$ is a local minm point for

$$f(x,y) = x^2+y^2 \quad s.t \quad x+2y=a.$$

$$f_1' = 2x, \quad f_{11}'' = 2, \quad f_{12}'' = 0$$

$$f_2' = 2y, \quad f_{21}'' = 0, \quad f_{22}'' = 2$$

$$g_1' = 1, \quad g_{11}'' = 0, \quad g_{12}'' = 0$$

$$g_2' = 2, \quad g_{21}'' = 0, \quad g_{22}'' = 0$$

$$D(a/5, 2a/5) = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 2 \end{vmatrix} = -10 < 0$$

$\Rightarrow$ local minm pt.

General Langrangean Problems (n variables & 1 equality constraint)

$$\max.(\min.)\ f(x_1, \ldots, x_n)$$
$$s.t \quad g(x_1, \ldots, x_n) = c$$

$$\mathcal{L}(x_1, \ldots, x_n) = f(x_1, \ldots, x_n) - \lambda(g(x_1, \ldots, x_n) - c)$$

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Solve to find $x_1, \ldots, x_n$ & λ

$$\begin{cases} \mathcal{L}_1'(x_1, \ldots, x_n) = f_1'(x_1, \ldots, x_n) - \lambda g_1'(x_1, \ldots, x_n) = 0 \\ \vdots \\ \mathcal{L}_n'(x_1, \ldots, x_n) = f_n'(x_1, \ldots, x_n) - \lambda g_n'(x_1, \ldots, x_n) = 0 \\ g(x_1, \ldots, x_n) = c \end{cases}$$

Economic Interpretation of λ (Only 1 constraint case)

We assume that $x_1^* = x_1^*(c), \ldots, x_n^* = x_n^*(c)$, $\lambda = \lambda(c)$

$$\therefore \underbrace{f^*(c)}_{\text{Optimal value function.}} = f\left(x_1^*(c), \ldots, x_n^*(c)\right)$$

$\boxed{\text{Results}}$ ① $\dfrac{d f^*(c)}{dc} = \lambda(c)$

② If dc is small, $\Delta f^*(c) \approx d f^*(c)$

$$f^*(c+dc) - f^*(c) \approx \lambda(c)\, dc$$

Envelope Results

$$\max_{x} \; f(x, r)$$

$$\text{s.t} \quad g(x, r) = 0$$

where $x = (x_1, \ldots, x_n)$

$r = (r_1, \ldots, r_k)$ vector of parameters (~~held~~ kept constant during maximization)

Values of $x_1, \ldots, x_n$ that maximize $f(x, r) \rightarrow x_1^*(r), \ldots, x_n^*(r)$

$$\underbrace{f^*(r)}_{\text{Optimal Value function}} = f\left(x^*(r), r\right) = f\left(x_1^*(r), \ldots, x_n^*(r), r_1, \ldots, r_k\right)$$

$$\mathcal{L}(x, r) = f(x, r) - \lambda g(x, r)$$

Envelope Th^m $\rightarrow$ $\boxed{\dfrac{\partial F^*(r)}{\partial r_j} = \dfrac{\partial \mathcal{L}(x^*(r), r)}{\partial r_j}}$ $(j = 1, \ldots, k)$

Q- max. $U(x,y) = 10\sqrt{x}\sqrt{y}$
s.t $px + qy = m$

(i) Find the opt. values of x & y as functions of p, q & m.
(ii) Find the opt. value function $U^*(p,q,m)$.
(iii) Find. $\partial U^*/\partial m$ & interpret it.

Let $U^*(p, q, m)$ be the optimal value function. Verify the envelope theorem.

(iii) $\dfrac{\partial U^*}{\partial m} = 5p^{-1/2}q^{-1/2} = \lambda$ (MU of money) ($\uparrow$ in max. uu' from $\uparrow$ng income by 1 unit)

$$\mathcal{L}(x, y, p, q, m) = 10\sqrt{x}\sqrt{y} - \lambda(px + qy - m)$$

$$\mathcal{L}_1' = \frac{5\sqrt{y}}{\sqrt{x}} - \lambda p = 0 \Rightarrow \frac{5\sqrt{y}}{\sqrt{x}p} = \lambda$$

$$\mathcal{L}_2' = \frac{5\sqrt{x}}{\sqrt{y}} - \lambda q = 0 \Rightarrow \frac{5\sqrt{x}}{\sqrt{y}q} = \lambda$$

$$\frac{\cancel{5}\sqrt{y}}{\sqrt{x}p} = \frac{\cancel{5}\sqrt{x}}{\sqrt{y}q}$$

$$px = qy \Rightarrow y = \frac{px}{q}$$

$$px + qy = m$$

$$2px = m \Rightarrow \boxed{x^* = m/2p}$$

$$\boxed{y^* = m/2q}$$

$$\boxed{\lambda = 5p^{-1/2}q^{-1/2}}$$

$$U^*(p,q,m) = 10\sqrt{x^*}\sqrt{y^*} = 10\sqrt{\frac{m}{2p}}\sqrt{\frac{m}{2q}} = 5p^{-1/2}q^{-1/2}m$$

① $\frac{\partial U^*(p,q,m)}{\partial p} = \frac{\partial \mathcal{L}(x^*, y^*, p, q, m)}{\partial p}$

$\frac{\partial U^*(p,q,m)}{\partial p} = -\frac{5}{2} p^{-3/2} q^{-1/2} m$ ✓

$\mathcal{L}(x^*, y^*, p, q, m) = 10\sqrt{x^*}\sqrt{y^*} - \lambda(px^* + qy^* - m)$

$\frac{\partial \mathcal{L}}{\partial p} = -\lambda x^* = -5p^{-1/2} q^{-1/2} \frac{m}{2p} = -\frac{5}{2} p^{-3/2} q^{-1/2} m$ ✓

② $\frac{\partial U^*(p,q,m)}{\partial q} = \frac{\partial \mathcal{L}(x^*, y^*, p, q, m)}{\partial q}$

③ $\frac{\partial U^*(p,q,m)}{\partial m} = \frac{\partial \mathcal{L}(x^*, y^*, p, q, m)}{\partial m}$

(*d*) An individual has a utility function $U = (x_1 x_2)^2$ based on the purchase of two commodities, $x_1 > 0$, $x_2 > 0$. The prices of goods x_1 and x_2 are Rs. 2 per unit and Rs. 4 per unit respectively while his total income is Rs. 400. If he tries to get the maximum utility within the budget, then find out the quantities that he should purchase to maximize his utility by using the Lagrange method. Check the SOC.

1)

$$Z = (x_1 x_2)^2 + \lambda\left[400 - 2x_1 - 4x_2\right] \quad (1)$$

$$\left.\begin{aligned} Z_1 &= 2x_1 x_2^2 - 2\lambda = 0 \\ Z_2 &= 2x_1^2 x_2 - 4\lambda = 0 \\ Z_\lambda &= 400 - 2x_1 - 4x_2 = 0 \end{aligned}\right\} \left(1\tfrac{1}{2}\right)$$

$$2x_1 x_2^2 = 2\lambda$$

$$2x_1^2 x_2 = 4\lambda$$

$$\therefore 2x_1 x_2^2 = 2 \cdot \frac{2x_1^2 x_2}{4}$$

$$x_1 x_2^2 = \frac{x_1^2 x_2}{2}$$

$$\boxed{2x_2 = x_1}$$

$$\therefore 400 = 2x_1 + 4x_2$$

$$= 2(2x_2) + 4x_2$$

$$= 8x_2$$

$$\therefore x_2 = 50$$

$$\therefore x_1 = 100 \quad (2)$$

$$x_1 = 100$$

$$x_2 = 50$$

$$\lambda = 25(10)^4$$

$$Z_{11} = 2x_2^2 = 0.5(10)^4$$

$$Z_{12} = 4x_1 x_2 = 2(10)^4$$

$$Z_{22} = 2x_1^2 = 2(10)^4$$

$$\therefore |\bar{H}| = \begin{vmatrix} 0 & 2 & 4 \\ 2 & 0.5(10)^4 & 2(10)^4 \\ 4 & 2(10)^4 & 2(10)^4 \end{vmatrix} \quad (1)$$

$$= 16(10)^4 > 0 \quad \left(\tfrac{1}{2}\right)$$

$\therefore$ utility is maximised at $(100, 50)$. (1)

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